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On a Simple Wave Approximation

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SUMMARY

An approximation (the linear version of Burgers' equation with appropriate initial data) to a simple wave initial value problem for a set of two linear coupled dissipative partial differential equations is discussed. It has been shown that for the class of square integrable initial functions of which the spectra (Fourier-transforms) have bounded support 2Δ the approximation is valid for some finite interval of time $[0, T(\Delta)]$. For some finite time $T_1 > T(\Delta)$ the approximation may fail. However, for $t\rightarrow\infty$, it is asymptotically valid again. For the class of initial conditions mentioned above expansions in series of the two solutions, which for every finite interval of time $[0, \tau]$ are convergent, may be constructed.

1. Introduction

1.1. *Statement of the Problem*

In physics one occasionally deals with the following simple wave initial value problem

$$\alpha(s, 0) = f(s)\,, \tag{1}$$

$$\beta(s, 0) = 0\,, \tag{2}$$

for the set of nonlinear dissipative partial differential equations

$$\alpha_t + [1+\varepsilon\Phi(\alpha, \beta)]\,\alpha_s = \mu(\alpha_{ss} - \beta_{ss})\,, \tag{3}$$

$$\beta_t - [1+\varepsilon\Psi(\alpha, \beta)]\,\beta_s = \mu(\beta_{ss} - \alpha_{ss})\,, \tag{4}$$

where s runs through the interval $(-\infty, \infty)$, t through $[0, \infty)$, $\Phi(\alpha, \beta)$ and $\Psi(\alpha, \beta)$ are continuous, often even monotonic functions of α and β, μ and ε are real positive constants and the subscripts s, t denote partial differentiation with respect to s, respectively t. Moreover Φ, Ψ and ε have been chosen such that if $\mu=0$ the remaining set is hyperbolic. A well-known example is found in Lighthill's theory of waves in a real gas (Lighthill [1]).

An exact and complete solution of this initial value problem is at present beyond all possibilities. Therefore Lighthill used an approximation. When $\mu=0$ it is seen that (4) is satisfied identically by $\beta(s, t)\equiv 0$. (3) then becomes a first order equation in α, which is readily solved. The resultant solution is a simple wave solution (cf. Lax [2]) for the hyperbolic set obtained by putting $\mu=0$. This explains the name we gave to the initial value problem. Lighthill's approximation is based on the assumption that, when μ is small, β will be negligible, at any rate for some finite interval of time. In this way one obtains from (3):

$$\alpha_t + [1+\varepsilon\Phi(\alpha, 0)]\,\alpha_s = \mu\alpha_{ss}\,, \tag{5}$$

which is an equation of *Burgers* type. In Lighthill's example Φ is linear in α. The exact solution of the initial value problem is known in that case. The approximation of the solution α of (1)–(4) by the solution α_0 of (1) and (5) henceforth will be called the simple wave approximation.

Now some questions that arise are:

1. May, for some finite interval of time $[0, T]$ and some $f(x)$, the simple wave approximation be used indeed? If this is true, what can be said about the dependence of T on the initial data? May T tend to infinity?

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2. Frequently in such problems one attempts an expansion in series of α where α_0 is the first term in the expansion. Does such an expansion really exist for some finite interval of time $[0, T]$ and some $f(x)$ and if so, does T depend on $f(x)$ and may T tend to infinity?

In general these questions would present rather formidable difficulties. Therefore we make a simplification by studying the linear system

$$\alpha_t + \alpha_s = \mu(\alpha_{ss} - \beta_{ss}), \tag{6}$$

$$\beta_t - \beta_s = \mu(\beta_{ss} - \alpha_{ss}), \tag{7}$$

subject to (1) and (2).

The Burgers approximation equation is given by

$$\alpha_t + \alpha_s - \mu\alpha_{ss} = 0. \tag{8}$$

The solution of (8) subject to (1) will be called α_0 again.

We still did not speak about what precisely we mean with a useful approximation. For solutions which are square integrable (we restrict ourselves to these solutions) we shall call the solution α_0 a useful approximation to α in the interval of time $[t_1, t_2]$ $(t_2 > t_1)$ if for every $t \in [t_1, t_2]$

$$\int_{-\infty}^{\infty} |\alpha - \alpha_0|^2 ds \ll \int_{-\infty}^{\infty} |\alpha|^2 ds.$$

$\int_{-\infty}^{\infty} |\alpha|^2 ds$ often has the meaning of the energy of the α-mode. It then provides a quite suitable norm for such a problem.

In a forthcoming paper we will treat a physical problem which leads to a special form of (3) and (4). In this case the equations can be transformed into linear equations of the form (6) and (7). Therefore the following considerations have at least some physical meaning.

2. Definitions and Notations

R: the interval $(-\infty, \infty)$ of the real numbers.

Q: a strip in the s–t plane containing all the points satisfying the inequalities $-\infty < s < \infty$ and $0 < t < T < \infty$.

Consider vector-valued functions of n complex-valued components $u = \text{col.}(u_1(s, t), \ldots, u_n(s, t))$ defined on R (t fixed) and Q respectively.

$L_2(R)$ is a Hilbert-space containing all square integrable n component vector-valued functions on R, with inner products (,) and norms $\| \ \|$ defined by

$$(u, v) = \int_{-\infty}^{\infty} u^\dagger(s) v(s) ds\,; \quad \|u\| = (u, u)^{\frac{1}{2}},$$

$u^\dagger$ being the hermitian transpose of u.

The Sobolev-space $W_2^m(R)$ (m a positive natural number) is a Hilbert-space containing all vector-valued $L_2(R)$ functions $u(s)$ whose generalised derivatives $D^k u$, $(k = 1, 2, \ldots, m)$ also are elements of $L_2(R)$ (Smirnow [3]). The inner product and norm are respectively

$$(u, v)_m = \sum_{i=1}^{m} (D^i u, D^i v) + (u, v); \quad \|u\|_m = (u, u)_m^{\frac{1}{2}}.$$

$C(R)$ is the set of all continuous, $C^i(R)$ the set of all i times continuously differentiable functions on R.

$L_2^\Delta(R)$ is a Hilbert-space containing all vector-valued functions $u(s)$ in $L_2(R)$, of which the Fourier-transform $\bar{u}(k)$, defined by

$$\bar{u}(k) = \int_{-\infty}^{\infty} u(s) \exp(-iks) \cdot ds, \tag{1}$$

vanishes identically outside a finite interval $[-\Delta, \Delta]$ $(\Delta \in R)$, with inner products $(\ , \ )_{R,\Delta}$ and norms $\| \ \|_{R,\Delta}$ defined by:

$$(u, v)_{R,\Delta} = \int_{-\infty}^{\infty} u^\dagger(s) v(s) ds; \ \|u\|_{R,\Delta} = (u, u)^{\frac{1}{2}}_{R,\Delta} .$$

Using Parseval's theorem one easily finds

$$\|u\|^2_{R,\Delta} = \frac{1}{2\pi} \int_{-\Delta}^{\Delta} \bar{u}^\dagger(k) \bar{u}(k) dk .$$

$L^\Delta_{2,q}(Q)$ is a Hilbert space containing all vector-valued square integrable functions on Q, with the properties: The Fourier-transforms defined similar to (1), for almost every $t \in [0, T]$ vanish identically outside a finite interval $[-\Delta, \Delta]$, $\Delta \in R$. The inner products $(\ , \ )_{Q,q,\Delta}$ and norms $\| \ \|_{Q,q,\Delta}$ are defined by

$$(u, v)_{Q,q,\Delta} = \frac{1}{2\pi} \int_0^T \int_{-\Delta}^{\Delta} q^2(k, t) \bar{u}^\dagger(k, t) \bar{v}(k, t) dk dt; \ \|u\|^2_{Q,q,\Delta} = (u, u) ,$$

where q is a positive continuous function of k and t, which for every $k \in [-\Delta, \Delta]$, $t \in [0, T]$ is bounded from above and from below. If $q = 1$, we simply write $L^\Delta_2(Q)$, $\| \ \|_{Q,\Delta}$ and $(\ , \ )_{Q,\Delta}$.

Finally we quote (for the proof see Smirnow [3], p. 486):

Lemma 1: *Let $u(s) \in W^m_2(R)$ then $D^p u(s) \to 0$ $(1 \leqq p < m)$ and $u(s) \to 0$ when $|s| \to \infty$.*

Remark: Where not stated otherwise all integrations are in the sense of Lebesque and all differentiations are meant in the generalised sense, although the classical notation will be retained.

3. The Solution of the Initial Value Problem

3.1 *Existence and Uniqueness*

Consider, for vector-valued functions of two components $u(s, t)$ defined on Q, the operator equation

$$u_t = \Lambda u , \tag{1}$$

where $u = \text{col.}(\alpha, \beta)$ and

$$\Lambda = D\frac{\partial}{\partial s} + A\frac{\partial^2}{\partial s^2} ,$$

$$D = \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}, \quad A = \mu \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} .$$

As Λ satisfies: (i) Λ is closed; (ii) for every $u \in D_\Lambda$ $\text{Re}(u, \Lambda u) \leqq \beta_1(u, u)$; (iii) for every $v \in D_{\Lambda^*}$ $\text{Re}(v, \Lambda^* v) \leqq \beta_2(v, v)$; (iv) $D_\Lambda = W^2_2(R)$ is dense in $L_2(R)$, where D_Λ is the domain of Λ, Λ^* the adjoint operator and β_1, β_2 are real positive constants, it can be proved that (de Graaf [4]):

Theorem 1

1. *The operator equation $u_t = \Lambda u$ is uniquely solvable for every $u(s, 0) \in W^n_2(R)$, $n \geqq 0$ $(W^0_2(R) \overset{\text{def}}{=} L_2(R))$ and for every $0 < t < T < \infty$ the solution is an element of $W^n_2(R)$.*
2. *$u(s, t) \to u(s, 0)$ for $t \to 0$ in the sense of the L_2-norm.*
3. *For an arbitrary initial condition $u(s, 0) \in W^n_2(R)$ $(n \geqq 0)$, $u(s, t)$ may be represented by*

$$u(s, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp -(Ak^2 + ikD)t \cdot \bar{f}(k) \exp(iks) \cdot dk , \tag{2}$$

where $\bar{f}(k)$ *is the Fourier-transform of the initial value* $u(s, 0)$ *and* $i=\sqrt{-1}$.
4. *The results of* (1), (2) *and* (3) *are true for* $L_2^4(R)$ *instead of* $L_2(R)$ *as well.*

For purely parabolic equations $u(s, 0) \in L_2(R)$ implies $u(s, t) \in W_2^n(R)$. The linear Burgers equation, to which the existence theorem applies in the same manner, belongs to this class of equations. The coupled system (1) however is not purely parabolic as A has an eigenvalue zero. This constitutes an essential difference between the coupled system and the linear Burgers equation satisfied by α_0.

3.2 *Stability and a Maximum-Modulus Principle*

Theorem 2. *Let* $u(s, 0) = f(s) \in L_2(R)$. *The solution of the system* (1) *is stable in the sense that for every* $t \geqq 0$

$$\|u(t)\| \leqq K \|f\|,$$

where K is a positive constant.

Proof: Let $f(s) \in L_2^4(R)$. Premultiply $u_t - Du_s - Au_{ss} = 0$ with $u^\dagger$, take the complex conjugate of the resulting equation and add. We find

$$\frac{\partial}{\partial t}(u^\dagger u) + \frac{\partial}{\partial s}(-u^\dagger Du - u^\dagger Au_s - u_s^\dagger Au) + 2u_s^\dagger Au_s = 0\,. \tag{3}$$

This is essentially the energy balance equation. If we integrate (3) along the entire s-axis and use lemma 1, we get

$$\|u(t)\| \leqq \|f\| \qquad (t \geqq 0)\,. \tag{4}$$

The remaining part of the proof depends on closure. As $L_2^4(R)$ is dense everywhere in $L_2(R)$ it is possible to find a sequence $\{f_n\} \subset L_2^4(R)$ which converges to $f \in L_2(R)$ in the sense of the L_2-norm. Then the solutions $u_n(s, t)$ corresponding to $f_n(s)$ also converge in that norm and according to (4) $\lim_{n\to\infty} u_n(s, t) = u(s, t)$, $u(s, 0) = f(s)$, $u(s, t)$ is a solution and $\|u(t)\| \leqq \|f\|$. This proves the theorem.

From a physical point of view it often is desirable or even necessary to have a maximum-modulus principle. It is given by:

Theorem 3. *Let* $u(s, 0) = f(s) \in W_2^1(R)$ *and* $\|f\|_1 \leqq \delta\sqrt{2}$ (δ *is some real positive constant*), *then we may define a function* $\tilde{u}(s, t) \in C(R)$ *such that for* $t \geqq 0$

$$\tilde{u}(s, t) = u(s, t) \quad \text{a.e.},$$

$$\sup_{s\in R} |\tilde{u}| \leqq \delta\,,$$

where $|\tilde{u}| = |\tilde{\alpha}| + |\tilde{\beta}|$.

Proof: Let $u(s, 0) = f(s) \in L_2^4(R)$ and $\|f\|_1 \leqq \delta\sqrt{2}$. If u is a solution, u_s is too. In this way we find the balance-equation (3) where u has been replaced by u_s. Adding (3) and the new equation, integrating the result along the entire s-axis and using lemma 1, we obtain

$$\|u(t)\|_1 \leqq \|f\|_1 = \delta\sqrt{2} \qquad (t \geqq 0)\,. \tag{5}$$

By means of closure we may show that (5) also holds for $f(s) \in W_2^1(R)$. From Sobolev's embedding theorem (Peletier [5]) we deduce the existence of a positive number M and a function $\tilde{u}(s, t) \in C(R)$ such that for $t \geqq 0$

$$\tilde{u}(s, t) = u(s, t) \qquad \text{a.e.},$$

$$\sup_{s\in R} |\tilde{u}(t)| \leqq M \|u(t)\|_1\,.$$

According to the before mentioned paper [5] the lowest value of M that may be chosen equals $\frac{1}{2}\sqrt{2}$, which completes the proof of the theorem.

3.3 *The Solution of the Simple Wave Initial Value problem*

The solution (2) may be written as

$$\alpha(s, t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} 1 + \int_{-\infty}^{\infty} 2\right] g^{(2)}(z) \exp h(z, \xi)t \cdot dz\,, \tag{6}$$

$$\beta(s, t) = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} 1 + \int_{-\infty}^{\infty} 2\right] g^{(1)}(z) \exp h(z, \xi)t \cdot dz\,, \tag{7}$$

where

$$\mu k = z\,, \quad \xi = \frac{s}{t}\,,$$

$$g^{(1)}(z) = (2i\mu)^{-1} z(1-z^2)^{-\frac{1}{2}} \bar{f}(\mu^{-1}z)\,,$$

$$g^{(2)}(z) = (2\mu)^{-1}\left[-1+(1-z^2)^{\frac{1}{2}}\right](1-z^2)^{-\frac{1}{2}} \bar{f}(\mu^{-1}z)\,,$$

$$h(z, \xi) = \mu^{-1}\left[iz(1-z^2)^{\frac{1}{2}} - z^2 + iz\xi\right].$$

The number 1 respectively 2 through the integration symbol means integration in the first, respectively, second sheet of the complex z-plane. The first sheet is defined by

$$\lim_{|z|\to\infty} \frac{(1-z^2)^{\frac{1}{2}}}{z} = -i \qquad 0 \leqq \arg z \leqq \pi\,,$$

and the second by

$$\lim_{|z|\to\infty} \frac{(1-z^2)^{\frac{1}{2}}}{z} = i \qquad 0 \leqq \arg z \leqq \pi\,.$$

This corresponds to cutting the z-plane from $-\infty$ to -1 and from 1 to ∞. In the remaining part of this paper we shall confine ourselves almost always to initial data in $L_2^4(R)$, although most of the results also apply to other classes of functions.

4. A Series Expansion of the Solution

Let $f(s)$ belong to $L_2^4(R)$. Defining the operators M and N by

$$M = \frac{\partial}{\partial t} + \frac{\partial}{\partial s} - \mu\frac{\partial^2}{\partial s^2}\,, \qquad N = \frac{\partial}{\partial t} - \frac{\partial}{\partial s} - \mu\frac{\partial^2}{\partial s^2}\,,$$

we find by applying the first one to (1.6) and the second to (1.7)

$$L\alpha = \mu^2 \frac{\partial^4 \alpha}{\partial s^4}\,, \tag{1}$$

$$L\beta = \mu^2 \frac{\partial^4 \beta}{\partial s^4}\,, \tag{2}$$

where

$$L = MN = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial s^2} - 2\mu\frac{\partial^3}{\partial s^2 \partial t} + \mu^2\frac{\partial^4}{\partial s^4}\,.$$

The initial conditions for this system are given by

$$\alpha(s, 0) = f(s)\,, \tag{3}$$

$$\alpha_t(s,0) = -\frac{df}{ds} + \mu\frac{d^2f}{ds^2}, \tag{4}$$

$$\beta(s,0) = 0, \tag{5}$$

$$\beta_t(s,0) = -\mu\frac{d^2f}{ds^2}. \tag{6}$$

As we assumed α and β to be in $L_2^\Delta(R)$, all the operations were allowed indeed. By now it may be seen that (1)–(6) are equivalent to (1.6) and (1.7) subject to (1.1) and (1.2). Consider in $L_2^\Delta(Q)$ the integral equations:

$$\alpha = B\alpha + \alpha_0, \quad \beta = B\beta + \beta_0, \tag{7}$$

with hold for every $t\in[0, T]$ and almost every $s\in R$ and where

$$B\alpha = \frac{\mu^2}{2\pi}\int_{-\Delta}^{\Delta} dk \int_0^t d\tau\, k^3 \sin k(t-\tau)\, e^{-\mu k^2(t-\tau)-iks}\bar{\alpha}(k,\tau),$$

$$\alpha_0 = \frac{1}{2\pi}\int_{-\Delta}^{\Delta} \bar{f}(k)\, e^{ik(s-t)-\mu k^2 t}\, dk, \tag{8}$$

$$\beta_0 = \frac{\mu}{2\pi}\int_{-\Delta}^{\Delta} k\bar{f}(k)\sin kt\, e^{iks-\mu k^2 t}\, dk.$$

Using these integral equations an expansion in series of the solution of our original system (1.6) and (1.7) will be derived. Of course other integral equations could have been used. However we choose the present ones as α_0 satisfies Burgers' equation and the initial value $\alpha(s,0)=f(s)$ exactly.

Theorem 4. *For every finite Δ and every finite positive number T, (7) has a solution which for every $t\in[0, T]$ belongs to $L_2^\Delta(R)$. It is the limit of the sequence*

$$\sum_{n=0}^{N} \alpha^{(n)} \quad \textit{for} \quad N\to\infty,$$

where

$$\alpha^{(0)} = \alpha_0, \quad \alpha^{(n+1)} = B\alpha^{(n)} \qquad (n = 0, 1, 2, \ldots).$$

Proof: From

$$\left|\sum_{n=1}^{\infty} \bar{\alpha}^{(n)}(k,t)\right| =$$

$$= \left|\sum_{n=1}^{\infty}\int_0^t d\tau_1 \ldots \int_0^{\tau_{n-1}} d\tau_n\, \mu^{2n} k^{3n} \prod_{j=1}^{n} \sin k(\tau_{j-1}-\tau_j)\cdot e^{-\mu k^2 t - ik\tau_n}|\bar{f}(k)|\right|$$

$$\leqq e^{-\mu k^2 t}|\bar{f}(k)| \sum_{n=1}^{\infty} \frac{t^n \mu^{2n}|k|^{3n}}{n!} = (e^{t\mu^2|k|^3}-1)\, e^{-\mu k^2 t}|\bar{f}(k)|, \tag{9}$$

where $\tau_0 = t$, we immediately deduce

$$\left\|\sum_{n=1}^{\infty} \alpha^{(n)}(t)\right\|_{R,\Delta} \leqq (e^{t\mu^2\Delta^3}-1)\cdot \|\alpha^{(0)}(t)\|_{R,\Delta} \tag{10}$$

which implies that for all $t\in[0, T]$

$$\sum_{n=0}^{N} \alpha^{(n)}(t)$$

has a limit for $N \to \infty$ in the sense of the $L_2^\Delta(R)$ norm. The limit will be called α. Furthermore we have:

$$\left\| B\left(\sum_{n=0}^{N} \alpha^{(n)} - \alpha \right) \right\|_{Q,\Delta} \leqq \mu^2 \Delta^3 T \left\| \sum_{n=0}^{N} \alpha^{(n)} - \alpha \right\|_{Q,\Delta} \qquad (t \in [0, T]).$$

For every $t \in [0, T]$ and almost every s

$$\sum_{n=0}^{N} \alpha^{(n)} = B \sum_{n=0}^{N-1} \alpha^{(n)} + \alpha_0 ,$$

which implies, using $\lim_{N\to\infty} \left\| \alpha - \sum_{n=0}^{N} \alpha^{(n)} \right\|_{Q,\Delta} = 0$, $\lim_{N\to\infty} \left\| B\left(\alpha - \sum_{n=0}^{N} \alpha^{(n)} \right) \right\|_{Q,\Delta} = 0$:

$$\| \alpha - B\alpha - \alpha_0 \|_{Q,\Delta} = 0 .$$

However as, according to (9), α is a continuous function of t, we obtain that for every $t \in [0, T]$ and almost every $s \in R$ α satisfies (7).

Theorem 5. *The solution $\alpha(s, t)$, found in theorem 4, for every $t \in [0, T]$ is unique in the sense of the L_2^Δ-norm and*

$$\| \alpha - \alpha_0 \|_{R,\Delta} \leqq (e^{T\mu^2\Delta^3} - 1) \| \alpha \|_{R,\Delta} \qquad (t \in [0, T]).$$

Proof: Let α' be another solution belonging to $L_2^\Delta(Q)$ as well.

Introduce the function $q(k, t)$ by

$$q^2(k, t) = \exp - \{ \mu^2 k^2 (e^{2\mu k^2 t} - 1) \} .$$

Call the difference $\alpha - \alpha' = \hat{\alpha}$. Using Schwarz's inequality we find:

$$\| B\hat{\alpha} \|_{Q,\Delta,q}^2 =$$

$$= \frac{\mu^4}{2\pi} \int_0^T dt \int_{-\Delta}^{\Delta} dk\, k^6 q^2(k, t) \left| \int_0^t \sin k(t-\tau) \cdot e^{-\mu k^2 (t-\tau)} q(k, \tau) \frac{\bar{\hat{\alpha}}(k, \tau)}{q(k, \tau)} d\tau \right|^2$$

$$\leqq \underset{\{t \in [0, T];\ k \in [-\Delta, \Delta]\}}{\text{maximum}} \left\{ \mu^4 k^6 \int_0^t dt \int_0^t d\tau \frac{q^2(k, t)}{q^2(k, \tau)} e^{-2\mu k^2 (t-\tau)} \right\} \cdot \| \hat{\alpha} \|_{Q,\Delta,q}^2$$

$$\leqq \tfrac{1}{2} \| \hat{\alpha} \|_{Q,\Delta,q}^2 .$$

As $\hat{\alpha} = B\hat{\alpha}$ for all $t \in [0, T]$ and almost every $s \in R$ and $0 = \| \hat{\alpha} - B\hat{\alpha} \|_{Q,\Delta,q}^2 \geqq \frac{1}{2} \| \hat{\alpha} \|_{Q,\Delta,q}^2$ we infer that $\| \hat{\alpha} \|_{Q,\Delta,q} = 0$ and so $\| \hat{\alpha} \|_{Q,\Delta} = 0$. Using the continuity with respect to t of $\hat{\alpha}$, this proofs the first part of the theorem. The second part follows immediately from (10) and the relationship $\| \alpha_0 \|_{R,\Delta} \leqq \| \alpha \|_{R,\Delta}$ which will be proved in the next section.

Corollary 1

It is clear that similar results may be proved for the β-mode by using

$$\sum_{n=1}^{\infty} \beta^{(n)}, \quad \text{where} \quad \beta^{(1)} = \beta_0 , \quad \beta^{(n+1)} = B\beta^{(n)} \qquad (n = 1, 2, 3, \ldots)$$

Corollary 2

For functions $f(s) \in L_2(R)$, but not in any $L_2^\Delta(R)$, similar theorems may be proved (the integration-interval with respect to k then runs from $-\infty$ to ∞) if $\bar{f}(k)$ tends to zero at least as fast as $\exp(-c|k|^3)$ $(c \geqq \delta > 0)$ when $|k| \to \infty$. This may be seen from (9).

It remains to be proved that α and β thus found also satisfy the original differential equations (1.6) and (1.7), subject to the initial conditions $\alpha(s, 0) = f(s)$, $\beta(s, 0) = 0$. Using the formulas of appendix 1, it is easily shown that they satisfy (1)–(6) but then we immediately may deduce that they satisfy the original equations and initial conditions as well.

5. Behaviour when $t \to \infty$

5.1 *The Simple Wave Approximation when $t \to \infty$*

At first we note an interesting relation between the energy of the α-mode and the energy of the β-mode.

Theorem 6. *Let $f(s) \in L_2(R)$, then for all $t \geqq 0$*

$$\|\alpha(t)\|^2 = \|\beta(t)\|^2 + \|\alpha_0(t)\|^2 , \tag{1}$$

where $\|\alpha_0\|^2$ is the energy of the solution α_0 of the corresponding Burgers problem.

Proof: Using (4.8), Parseval's theorem and transformation to the integration variable z by means of $z = \mu k$ gives

$$\|\alpha_0(t)\|^2 = \frac{1}{2\pi\mu} \int_{-\infty}^{\infty} e^{-2z^2\mu^{-1}t} |\bar{f}(\mu^{-1}z)|^2 dz . \tag{2}$$

As is easily seen from (3.6) and (3.7) the solutions α and β may be represented by one integral (with respect to z) each. Transforming the integration variable z to k, using Parseval's theorem and transforming backwards, we find that

$$\|\beta(t)\|^2 = \frac{1}{2\pi\mu} \int_{-\infty}^{\infty} \frac{z^2}{1-z^2} \sin^2\left(z\sqrt{1-z^2}\,\frac{t}{\mu}\right) \cdot e^{-2z^2\mu^{-1}t} |\bar{f}(\mu^{-1}z)|^2 dz , \tag{3}$$

and $\|\alpha(t)\|^2$ equals the sum of $\|\alpha_0(t)\|^2$ and $\|\beta(t)\|^2$.

Remark: Until now we have not been able to prove this relationship without using the integral-representations of the solutions. An alternative proof is based on the remark that, using the notation of 3.1,

$$\|\alpha\|^2 - \|\beta\|^2 = -(u, Du) .$$

This can be worked out, using the representation (3.2). In this way one is led to (1) provided that the second component of $\bar{f}(k)$ in (3.2) vanishes. The main theorem of this chapter is given by:

Theorem 7. Let $f(s) \in L_2^4(R)$, $\bar{f}(k)$ analytic in a vicinity of $k = 0$ and $\bar{f}(0) \neq 0$. Then a constant K exists such that for $t \to \infty$

$$\|\alpha - \alpha_0\|^2 \leqq K t^{-1} \|\alpha\|^2 .$$

Proof: The proof will be split into some lemmas.

Lemma 2. *Let $f(s) \in L_2(R)$. For every $t \geqq 0$*

$$\|\alpha - \alpha_0\|^2 = 2\|\alpha\|^2 + \|\beta\|^2 + I ,$$

where

$$I = \frac{1}{2\pi}\left[\int_{-\infty}^{\infty} 1 + \int_{-\infty}^{\infty} 2\right] \phi(z) \left[e^{\Psi_a(z)t} + e^{\Psi_b(z)t}\right] dz ,$$

$$\phi(z) = \tfrac{1}{2}\left[1 - (1-z^2)^{\frac{1}{2}}\right](1-z^2)^{-\frac{1}{2}} |\bar{f}(\mu^{-1}z)|^2 ,$$

$$\Psi_a(z) = \mu^{-1}\left[iz + iz(1-z^2)^{\frac{1}{2}} - 2z^2\right] , \quad \Psi_b(z) = \Psi_a(-z) .$$

Proof: Similar to the proof of theorem 6.

In the three following lemmas we assume $\bar{f}(k)$ to satisfy the conditions of theorem 7.

Lemma 3. *A real positive constant L exists such that when $t\to\infty$*

$$\|\alpha_0(t)\|^2 \geqq Lt^{-\frac{1}{2}}.$$

Proof: Using the method of saddle-points (de Bruijn [6]) one easily finds

$$\|\alpha_0\|^2_{R,\Delta} = \frac{|\bar{f}(0)|^2}{2\pi\mu}\left\{\frac{\pi\mu}{2t}\right\}^{\frac{1}{2}} + O(t^{-\frac{3}{2}}) \qquad (t\to\infty),$$

from which the lemma immediately follows.

Lemma 4. *Real and positive constants K_1 and K_2 exist such that when $t\to\infty$*

$$\|\beta\|^2 \leqq K_1 t^{-\frac{3}{2}}, \tag{4}$$

$$\left|\|\alpha\|^2 - \frac{|\bar{f}(0)|^2}{2\mu\pi}\left\{\frac{\pi\mu}{2t}\right\}^{\frac{1}{2}}\right| \leqq K_2 t^{-\frac{3}{2}}.$$

Proof: At first we remark that if the points -1 and 1 are contained in the integration interval the integrals

$$\left[\int_{1-\varepsilon}^{1+\delta} + \int_{-1-\delta}^{-1+\varepsilon}\right]\frac{z^2}{1-z^2}\sin^2\left(z\sqrt{1-z^2}\,\frac{t}{\mu}\right)\cdot e^{-2z^2\mu^{-1}t}|\bar{f}(\mu^{-1}z)|^2\,dz,$$

where $\varepsilon>0$, $\delta>0$, for $t\to\infty$ are $O(e^{-t/\mu}t^2)$ when ε and δ are chosen small enough. Let ε and δ be chosen in that way then

$$\|\beta(t)\|^2 = O(t^2 e^{-t/\mu}) + \frac{1}{2\pi}\left[\int_C 1 + \int_C 2\right]\frac{z^2}{4\mu(z^2-1)}|\bar{f}(\mu^{-1}z)|^2 e^{2h(z,0)t}\,dz$$

$$-\frac{1}{2\pi}\int_C \frac{z^2}{2\mu(z^2-1)}|\bar{f}(\mu^{-1}z)|^2 e^{-2z^2\mu^{-1}t}\,dz,$$

where, if $\mu\Delta > 1+\delta$

$$C = [-\mu\Delta, -1-\delta] + [-1+\varepsilon, 1-\varepsilon] + [1+\delta, \mu\Delta]$$

and if $\mu\Delta \leqq 1+\delta$

$$C = [-1+\varepsilon, 1-\varepsilon].$$

The real part of $h(z, 0)$ is smaller than $-(2\mu)^{-1}$ at the positive side of the cuts in both sheets of the z-plane. This implies that the integrals along $[-\mu\Delta, -1-\delta]$ and $[1+\delta, \mu\Delta]$ are $O(e^{-t/\mu})$ for $t\to\infty$. As $(\partial h/\partial z)(z, 0) \neq 0$ for $z\in[-1+\rho, 1-\rho]$ we find by using appendix 2 that the part of the first two integrals in the right-hand side running from $-1+\varepsilon$ to $1-\varepsilon$ is $O(t^{-2})$ when $t\to\infty$. The remaining integral from $-1+\varepsilon$ to $1-\varepsilon$ can be approximated by means of the saddle-point techniques which finally results in (4). The second part is also proved by using these techniques.

Lemma 5. *When $t\to\infty$*

$$I = -\frac{|\bar{f}(0)|^2}{\pi\mu}\left\{\frac{\pi\mu}{2t}\right\}^{\frac{1}{2}} + O(t^{-\frac{3}{2}}).$$

Proof: As in lemma 4 we may choose $\varepsilon>0$ and $\rho>0$ such that

$$\sum_{j=1}^{2}\left[\int_{1-\varepsilon}^{1+\delta} j + \int_{-1-\delta}^{-1+\varepsilon} j\right]\phi(z)\left[e^{\Psi_a(z)t} + e^{\Psi_b(z)t}\right]dz = O(t\,e^{-t/\mu})$$

when $t \to \infty$. The contributions of $[-\mu\Delta, -1-\rho]$ and $[1+\rho, \mu\Delta]$ (if there are any) are $O(\exp -(1+\frac{1}{2}\sqrt{3})\mu^{-1}t)$ when $t \to \infty$. Remain two integrals running from $-1+\varepsilon$ to $1-\varepsilon$. Using appendix 2 we find the integral defined in the first sheet to be $O(t^{-2})$. Application of the method of saddle-points to the second integral then yields the required result. We now return to the proof of our theorem. From lemma 2, 4 and 5 we deduce the existence of a real positive constant K such that when $t \to \infty$

$$\|\alpha - \alpha_0\|^2 \leq K t^{-\frac{3}{2}}.$$

From this relationship and lemma 3 the theorem immediately follows.

Remark 1. The condition $\bar{f}(0) \neq 0$ is not essential to the proof. If $\bar{f}(0) = 0$ the result turns out to be similar.

Remark 2. The proof may also be given for other classes of functions, for instance the Hermite functions

$$f(s) = (-1)^n e^{-\frac{1}{2}s^2} \frac{d^n}{ds^n}(e^{-s^2}) \qquad (n = 0, 1, 2, \ldots),$$

the "Laguerre functions"

$$f(s) = \begin{Bmatrix} s^n e^s & s \leq 0 \\ 0 & s > 0 \end{Bmatrix} \qquad (n = 0, 1, 2, \ldots) \tag{5}$$

and modulations of these functions with $\exp(ik_0 s)$. Only slight modifications have to be made.

Remark 3. Although the spectral range of the initial function $f(s)$ may be very large we see that when $t \to \infty$ Burgers' equation perfectly describes the behaviour of the α-mode. This rather surprising result is essentially due to the fact that the solution when $t \to \infty$ almost only depends on the spectrum $\bar{f}(k)$ of $f(s)$ in a vicinity of $k = 0$.

5.2. *The Asymptotic Behaviour of α and β when $t \to \infty$*

As the results concerning the α and β-mode are a bit surprising it seems worth while to look at the asymptotic behaviour of the solutions α and β itself and to see what actually is going on.

For this purpose we shall use the method of saddle-points again. These are located at the roots of $\partial h/\partial z = 0$ or

$$\xi = -2iz - \frac{(1-2z^2)}{(1-z^2)^{\frac{1}{2}}} = G(z).$$

It is clear that with the possible exception of a finite number of values of ξ $h(z, \xi)$ has three saddle-points. The reflection principle of Schwarz (Bieberbach [7]) shows that $G(iz) = G^*(iz^*)$ where z^* is the complex conjugate of z. If h has a saddle-point in iz then $G^*(iz) = G(iz)$ and so $G(iz) = G(iz^*)$ which implies that the saddle-points are located symmetrically with respect to the imaginary axis of the z-plane. As $G_1(z) = -G_2(-z)$ (the index defines the sheet of the z-plane the function is defined in), it follows that if the pair $\{\xi, z\}$ satisfies $\xi = G_1(z)$ then $\{-\xi, -z\}$ is a solution of $\xi = G_2(z)$. Therefore we can confine our investigation to $\xi \geq 0$. By now it may be seen quite easily that if ξ runs from 0 to ∞ h_2 has a saddle-point $z_1(\xi)$ running along the imaginary axis of the z-plane from $-i\infty$ to $i\infty$ and h_1 has two saddle-points $z_2(\xi)$ and $z_3(\xi)$ in the upper half plane. They are situated as shown in fig. 1.

By now it is quite standard to derive the asymptotic expansions of the integral-representations. We shall not go into the details of this procedure but merely sketch its result derived for the case we use the continuous functions of (5) as initial data. When $t \to \infty$ the α-mode of the wave-phenomenon consists of a right- and a left-travelling wave. It has sharp "peaks" around

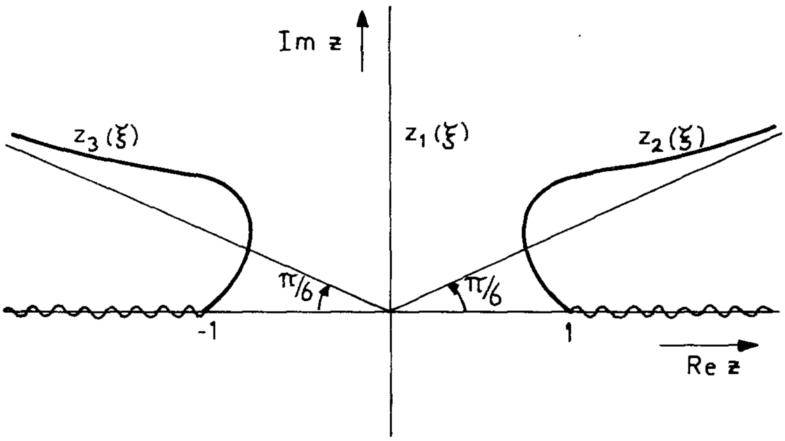

Fig. 1. The contours of the saddle-points $z_1(\xi)$, $z_2(\xi)$ and $z_3(\xi)$.

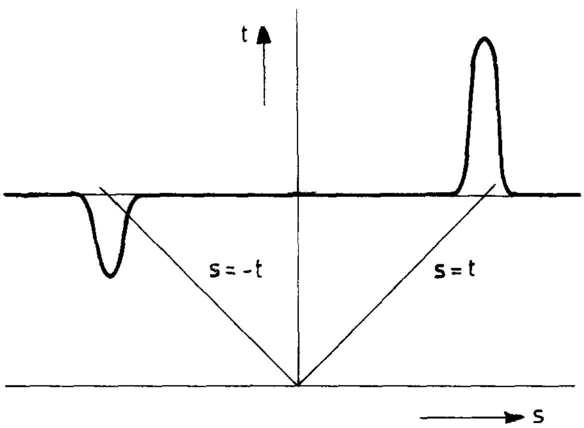

Fig. 2. The α-mode.

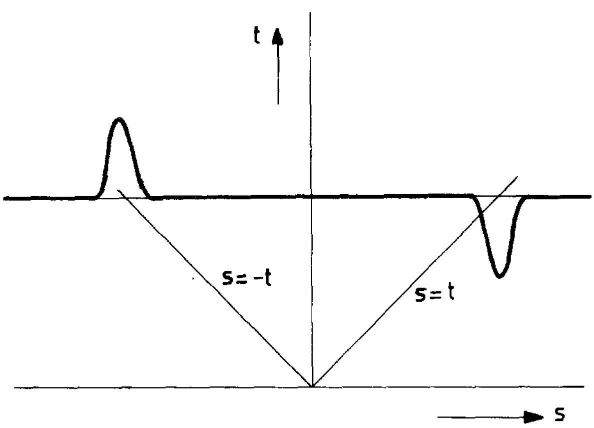

Fig. 3. The β-mode.

and maxima along $s=t$ (of $O(t^{-\frac{1}{2}})$) and $s=-t(O(t^{-\frac{3}{2}}))$. To the left and to the right of these maxima $\alpha=O(e^{-ct})$, $c>0$.

The β-mode has the same features as the α-mode, however both maxima are $O(t^{-\frac{3}{2}})$. The amplitudes at these maxima are opposite in sign. The width at half maximum of the peaks is $O(t^{-\frac{1}{2}})$. In fig. 2 and 3 the situation, when n is even, is drawn. For $L_2^4(R)$ and also for Hermite functions similar results may be derived.

6. Conclusions

For every $f(s) \in L_2^{\Delta}(R)$ we have shown that for every finite interval of time $[0, T]$ a convergent expansion in series of the solutions α and β does exist indeed. The first term in the expansion of α is given by α_0, the solution of the Burgers approximation problem. Furthermore for every $t \in [0, T]$

$$\|\alpha - \alpha_0\|_{R,\Delta} \leqq (e^{\Delta^3 \mu^2 T} - 1) \|\alpha\|_{R,\Delta},$$

which shows that the approximation in the interval $[0, T]$ may be made as close as one wants to by choosing Δ, μ and T.

From (5.1), (5.2) and (5.3) some further information may be drawn. Let $f(s) = \pi^{-1}(\sin \mu^{-1} s)/s$, that is $\bar{f}(\mu^{-1} z) = 1$ for $|z| \leqq 1$ and $\bar{f}(\mu^{-1} z) = 0$ elsewhere on the real z-axis. Then from (5.3) we easily obtain for $t = \mu$

$$\|\beta\|^2 > \frac{9}{20\pi\mu} \int_{-1}^{1} z^4 e^{-2z^2} dz > \tfrac{1}{13} \|\alpha_0\|^2,$$

or using (5.1), (5.2) and the triangle inequality

$$\|\alpha - \alpha_0\| > \frac{1}{2\sqrt{3}} \|\alpha_0\|,$$

which implies that under circumstances the simple wave approximation may break down.

However, as we have seen before, when $t \to \infty$ a positive constant K exists such that

$$\|\alpha - \alpha_0\| \leqq K t^{-\frac{1}{2}} \|\alpha_0\|,$$

and so the approximation may be used again.

Appendices

Appendix 1

Define

$$P(k, t-\tau, s) = \frac{\mu^2 k^3}{2\pi} \sin k(t-\tau) \cdot \exp[-\mu k^2 (t-\tau) + iks].$$

Lemma 6. *For every $t \in [0, T]$ and almost every $s \in R$, the solution of* (4.7) *satisfies*:

$$\frac{\partial}{\partial t}(\alpha - \alpha_0) = \int_{-\Delta}^{\Delta} dk \int_0^t d\tau \, P_t(k, t-\tau, s) \bar{\alpha}(k, \tau),$$

$$\frac{\partial^2}{\partial t^2}(\alpha - \alpha_0) = \int_{-\Delta}^{\Delta} dk \int_0^t d\tau \, P_{tt}(k, t-\tau, s) \bar{\alpha}(k, \tau) + \int_{-\Delta}^{\Delta} dk \, P_t(k, 0, s) \bar{\alpha}(k, t),$$

$$\frac{\partial^{n+1}}{\partial s^n \partial t^j}(\alpha - \alpha_0) = \int_{-\Delta}^{\Delta} dk \int_0^t d\tau \frac{\partial^j}{\partial t^j} P(k, t-\tau, s) \bar{\alpha}(k, \tau)(ik)^n, \; j = 0, 1; \; n = 0, 1, 2.$$

Proof:

$$\int_0^t d\tau \int_{-\Delta}^{\Delta} dk \frac{\partial^j}{\partial t^j} P(k, t-\tau, s)(ik)^l \bar{\alpha}(k, \tau) \qquad (j = 0, 1, 2; \; l = 0, 1, 2, \ldots)$$

converges uniformly with respect to $s \in R$, $t \in [0, T]$ for

$$\int_{-\Delta}^{\Delta} dk \int_0^t d\tau \left| (ik)^l \frac{\partial^j}{\partial t^j} P(k, t-\tau, s) \bar{\alpha}(k, \tau) \right| \leqq \frac{\mu^2}{2\pi} \Delta^{l+3} (\Delta + \mu \Delta^2)^j \int_{-\Delta}^{\Delta} dk \int_0^T d\tau |\bar{\alpha}(k, \tau)|$$

$$\leqq \mu^2 \left(\frac{\Delta T}{\pi}\right)^{\frac{1}{2}} \Delta^{l+3} (\Delta + \mu \Delta^2)^j \|\alpha\|_{Q,\Delta}$$

and $\alpha \in L_2^\Delta(Q)$.

Now we have

$$\int_0^t d\tau \int_{-\Delta}^{\Delta} dk\, P_t(k, t-\tau, s)\,\bar{\alpha}(k,\tau) = \frac{\partial}{\partial t}\int_0^t d\theta \int_0^\theta d\tau \int_{-\Delta}^{\Delta} dk\, P_\theta(k, \theta-\tau, s)\,\bar{\alpha}(k,\tau)$$

$$= \frac{\partial}{\partial t}\int_{-\Delta}^{\Delta} dk \int_0^t d\tau \int_\tau^t d\theta\, P_\theta(k, \theta-\tau, s)\,\bar{\alpha}(k,\tau)$$

$$= \frac{\partial}{\partial t}\int_{-\Delta}^{\Delta} dk \int_0^t d\tau\, [P(k, t-\tau, s) - P(k, 0, s)]\,\bar{\alpha}(k,\tau) =$$

$$= \frac{\partial(\alpha-\alpha_0)}{\partial t}$$

for $P(k, 0, s) = 0$.

The other formulae can be proved in a similar way.

Appendix 2

Define

$$I = \int_{-N}^{N} g(z)\,e^{h(z)t}\,dz \qquad (t \geqq 0,\ N \geqq N_0 > 0)\,.$$

$g(z)$ and $h(z)$ are complex valued functions satisfying:

(a) For every z satisfying $N \geqq |z| \geqq \delta > 0$ a positive constant p exists such that

$$e^{h(z)t} = O(e^{-pt}) \qquad (t \to \infty)\,.$$

(b) For every $|z| \leqq N$, $t \geqq 0$

$$|e^{h(z)t}| \leqq 1\,.$$

(c) For every $|z| \leqq N$ $h(z)$ is three times continuously differentiable and $dh/dz \neq 0$.

(d) $\int_{-N}^{N} |g(z)|\,dz$ is finite.

(e) $g(z)$ is analytic in a vicinity of $z = 0$.

Lemma 7. *When* $t \to \infty$

$$I = O(t^{-2})\,.$$

Proof: Let $g(z)$ be analytic in an open interval containing $[-\varepsilon, \varepsilon]$ $(\varepsilon \geqq \delta > 0)$. As is easily seen using (a) and (d) a positive constant p exists such that

$$\left[\int_{-a}^{-\varepsilon} + \int_{\varepsilon}^{a}\right] g(z)e^{h(z)t}\,dz = O(e^{-pt})$$

when $t \to \infty$.

From partial differentiation of the remaining integral using (c), (e) and the analyticy of $g(z)$ one obtains

$$I = O(e^{-pt}) + \frac{\mu^2}{t^2}\int_{-\varepsilon}^{\varepsilon} \left\{\frac{g''h' - gh''}{(h')^3} - \frac{3(g'h' - gh'')}{(h')^4}\right\} e^{h(z)t}\,dz\,, \tag{1}$$

where the accent(s) denote differentiation(s).

(1) and (b) now immediately imply the theorem.

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